THE EFFECTS OF NONLOCALITY ON THE EVOLUTION OF HIGHER ORDER FLUXES IN NON-EQUILIBRIUM THERMODYNAMICS

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ABSTRACT. The role of gradient dependent constitutive spaces is investigated on the example of Extended Thermodynamics of rigid heat conductors. Different levels of nonlocality are developed and the different versions of extended thermodynamics are classified. The local form of the entropy density plays a crucial role in the investigations. The entropy inequality is solved under suitable constitutive assumptions. Balance form of evolution equations is obtained in special cases. Closure relations are derived on a phenomenological level.

1. INTRODUCTION

Weakly nonlocal thermodynamic theories are those that introduce the space derivatives of the basic variables into constitutive functions [1]. Second Law restricts considerably the form of the constitutive quantities and gives a genuine insight into the structure of the theories. Weakly nonlocal constitutive functions are mostly investigated in relation of material microstructure in mechanics [2, 3] or to find nonlocal extensions of classical theories [1].

In this paper we investigate nonlocal constitutive spaces with different levels of nonlocality, namely of different order of derivatives. However, we derive also the restrictions that are due to locality assumptions on different levels. In our analysis we assume a non-equilibrium entropy function that can be approximated by its values measured at the equilibrium. Such an assumption is referred to as *local state* hypothesis [4].

We restrict ourselves to extended thermodynamic theories of rigid heat conductors [5, 6, 7] and introduce the heat flux together with a second order tensor as internal variables. The balance structure of the evolution equations is not postulated. Furthermore, the entropy current is regarded as a constitutive quantity and we are to give a complete solution of the thermodynamic constraints i.e. both the equalities and the residual dissipation inequality.

In a previous work [8] the local theory has been developed in the details and the evolution equations for fluxes of higher tensorial order have been obtained. Also it was proved that under particular assumptions on the entropy density and the entropy current the balance form can be recovered. Moreover, the system of equations was closed, in that the evolution equations for the highest order variable in the hierarchy - ordinary differential equations - can be interpreted as a closure relation.

In the present paper we extend our investigation to the case of weakly nonlocal constitutive state spaces. The solutions are derived with the help of minimal assumptions on the form of the entropy, on its flux or on the evolution equations of the internal variables. In this way the different solutions of the entropy inequality are clearly classified. The most general assumption, that covers all existing phenomenological theories lies on the concept of *current multipliers*, which represent some constitutive functions entering the entropy current. We

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will see that the final evolution equations are more general than the traditional balance ones. The conditions to recover the classical cases are clarified.

In Section 2 we investigate first order nonlocality by applying Liu procedure [9, 10] for the exploitation of Second Law.

In Section 3 we solve the Liu equation in the case of local state and local evolution equations for the dynamic variables. These assumptions result in a set of rather unusual restrictions from which we argue that some kind of nonlocality, either of the constitutive space or of the evolution equations, seems to be unavoidable. On the other hand, if we face with nonlocal state but local evolution equations then the entropy current is local, provided that the entropy density does not depend on the gradient of the internal energy.

In Section 4 we investigate the traditional assumptions of Extended Irreversible Thermo-dynamics based on the following form of the local entropy [11]

$$(1.1) \qquad s(e, q_i, \Phi_{ij}) = s_0 - \frac{1}{2} m_{ij} q_i q_j - \frac{1}{2} n_{ijkl} \Phi_{ij} \Phi_{kl},$$

where the matrices m_{ij} and n_{ijkl} are constitutive functions and s_0 is the *equilibrium entropy*, that depends only on the internal energy. We first suppose the entropy current is given as [12]

$$(1.2) \qquad j_i = A_{ij} q_j + B_{ijk} \Phi_{jk},$$

where A_{ij} and B_{ijk} are constitutive functions, the so called *current multipliers*. Then we explore the less general case [13], too

$$(1.3) \qquad j_i = \frac{\partial s}{\partial e} q_i + \frac{\partial s}{\partial q_k} \Phi_{ki}$$

It is worth noticing that the general form of the entropy current (1.2) reduces to (1.3) when $A_{ij} = \frac{\partial s}{\partial e} \delta_{ij}$ and $B_{ijk} = \frac{\partial s}{\partial q_j} \delta_{ik}$. We investigate different assumptions that can be compatible with the balance form of the evolution equations.

In Section 5 we consider second order nonlocality but conserve the form (1.1) of the entropy density and the expression (1.2) of the entropy current. In such a case, due to the enlargement of the state space, the balance form can be obtained even if the general constitutive equation (1.2) holds true. We show that all previous examples can be recovered under simple special assumptions.

In Section 6 we point out some nonlocal effects arising in thermal wave propagation at low temperature, which are described by the celebrated Guyer-Krumhansl equation [14, 15, 16]. Such an equation has been derived by the authors by solving a linearized Boltzmann equation for phonon gas hydrodynamic. Here we prove that it can be obtained in the classical macroscopic framework of nonlocal irreversible thermodynamics.

The previous results are discussed in Section 7, where a table shows the connections between the constitutive assumptions and the thermodynamic restrictions, as far as the locality and nonlocality are concerned.

2. First order nonlocality - exploitation of the Second Law

In a rigid heat conductor at rest we start from the following local balance equation of the internal energy

$$(2.1) \qquad \dot{e} + q_{i'i} = 0,$$

where e is the density of internal energy, q_i $i = 1, 2, 3$ are the components of the heat flux, $\dot{f} \equiv \frac{\partial f}{\partial t}$, $f_{'i} \equiv \frac{\partial f}{\partial x_i}$, x_i $i = 1, 2, 3$ are the Cartesian coordinates of the points of the body and the Einstein convention of summation over the repeated indices has been applied. The only equilibrium variable will be the internal energy e, while the first dynamic variable is supposed to be the heat flux q_i. As a further dynamic variable let us choose a second order tensor, whose components will be denoted by Φ_{ij}, $i, j = 1, 2, 3$. Hence the *basic state space* (the

wanted fields) in our investigations will be spanned by the variables (e, q_i, Φ_{ij}). This is a 13 field theory, because the number of the independent fields is 13. However, some reductions are possible. For instance in Extended Thermodynamics tensor Φ_{ij} is identified with the symmetric momentum flux $N_{[ij]}$ coming from kinetic theory [7]. Then the unknown fields reduce to 10 and we face with a 10-field theory. Furthermore it is possible to decompose $N_{[ij]}$ into an isotropic part, which is related to the internal energy, and a deviatoric part according to the equation $N_{[ij]} = \frac{1}{3} e \delta_{[ij]} + N_{<ij>}$, where $N_{<ij>}$ is symmetric and traceless [17]. In such a particular case the internal energy coincides with one of the six independent components of $N_{[ij]}$ and we deal with a 9-field theory. We are investigating a first order weak nonlocality in all variables of the basic state, therefore the *constitutive space* is spanned by the basic state and its spacial derivatives, that is the fields $(e, q_i, \Phi_{ij}, e_{'i}, q_{i'j}, \Phi_{ij'k})$. We assume that the evolution equations for the heat current q_i and for Φ_{ij} can be written in the following rather general form

$$(2.2) \qquad \dot{q}_i = g_i,$$
$$(2.3) \qquad \dot{\Phi}_{ij} = f_{ij},$$

where g_i and f_{ij} are constitutive functions. With the assumption of first order nonlocality the spacial derivatives of the above equations give further restrictions [18, 19].

$$(2.4) \qquad \dot{e}_{'i} + q_{j'ji} = 0,$$
$$(2.5) \qquad \dot{q}_{i'j} - g_{i,j} = 0,$$
$$(2.6) \qquad \dot{\Phi}_{ij'k} - f_{ij'k} = 0.$$

These equations are sometimes referred to as *prolonged forms* of the evolution equations (2.1), (2.2) and (2.3).

The local balance of entropy is given by

$$(2.7) \qquad \dot{s} + j_{i'i} = \sigma_s,$$

with s standing for the entropy density, j_i $i = 1, 2, 3$ for the components of the entropy current and σ_s for the density of entropy production. Second Law of Thermodynamics forces σ_s to be nonnegative.

In the following we will investigate the restrictions from the inequality of the Second Law with the general assumption that both the entropy and the entropy flux are constitutive quantities. The method of the exploitation is given by the Liu procedure [9]. However, according to our calculations, in the present case a generalized Coleman-Noll [20] procedure gives identical results.

Let us introduce the Lagrange-Farkas multipliers [9, 10] Γ^1, Γ_i^2 and Γ_{ij}^3 for the evolution equations (2.1), (2.2) and (2.3) respectively. The multipliers Γ_i^4, Γ_{ij}^5 and Γ_{ijk}^6 are for the prolonged evolution equations (2.4), (2.5) and (2.6) respectively.

Now, Liu procedure gives

$$
\begin{aligned}
\partial_1 s \dot{e} \quad + \quad & (\partial_2 s)_i \dot{q}_i + (\partial_3 s)_{ij} \dot{\Phi}_{ij} + (\partial_4 s)_i \dot{e}_{'i} + (\partial_5 s)_{ij} \dot{q}_{i'j} + \\
+ (\partial_6 s)_{ijk} \dot{\Phi}_{ij'k} \quad + \quad & (\partial_1 j_i) e_{'i} + (\partial_2 j_i)_j q_{j'i} + (\partial_3 j_i)_{jk} \Phi_{jk'i} + (\partial_4 j_i)_j e_{'ij} + \\
+ (\partial_5 j_i)_{jk} q_{j'ki} \quad + \quad & (\partial_6 j_i)_{jkl} \Phi_{jk'li} - \\
- \Gamma^1 \left(\dot{e} + q_{i'i} \right) \quad - \quad & \Gamma_i^2 \left(\dot{q}_i - g_i \right) - \Gamma_{ij}^3 \left(\dot{\Phi}_{ij} - f_{ij} \right) - \Gamma_i^4 \left(\dot{e}_{'i} + q_{j'ji} \right) - \\
- \Gamma_{ij}^5 \big(\dot{q}_{i'j} \quad - \quad & (\partial_1 g_i) e_{'j} - (\partial_2 g_i)_k q_{k'j} - (\partial_3 g_i)_{kl} \Phi_{kl'j} - (\partial_4 g_i)_k e_{'kj} - \\
\quad - \quad & (\partial_5 g_i)_{kl} q_{k'lj} - (\partial_6 g_i)_{klm} \Phi_{kl'lmj} \big) \\
- \Gamma_{ijk}^6 \big(\dot{\Phi}_{ij'k} \quad - \quad & (\partial_1 f_{ij}) e_{'l} - (\partial_2 f_{ij})_l q_{l'k} - (\partial_3 f_{ij})_{lm} \Phi_{lm'k} - (\partial_4 f_{ij})_l e_{'lk} - \\
\quad - \quad & (\partial_5 f_{ij})_{lm} q_{l'mk} - (\partial_6 f_{ij})_{lmn} \Phi_{lm'nk} \big) \geq 0.
\end{aligned}
$$

Here ∂_n, $n = 1, 2, 3, 4, 5, 6$ denotes the partial derivatives of the constitutive functions according to the variables $(e, q_i, \Phi_{ij}, e\prime_i, q_{i\prime j}, \Phi_{ij\prime k})$ respectively. After some rearrangements of the inequality one obtains the Lagrange-Farkas multipliers from the first set of the Liu equations. These are obtained by imposing the coefficients of the time derivatives vanish.

$$(2.8) \qquad\qquad \partial_1 s = \Gamma^1,$$

$$(2.9) \qquad\qquad (\partial_2 s)_i = \Gamma^2_i,$$

$$(2.10) \qquad\qquad (\partial_3 s)_{ij} = \Gamma^3_{ij},$$

$$(2.11) \qquad\qquad (\partial_4 s)_i = \Gamma^4_i,$$

$$(2.12) \qquad\qquad (\partial_5 s)_{ij} = \Gamma^5_{ij},$$

$$(2.13) \qquad\qquad (\partial_6 s)_{ijk} = \Gamma^6_{ijk}.$$

The second set of Liu equations is obtained by taking equal to zero the multipliers of the second order space derivatives. By applying (2.8)-(2.13) one can write them as

$$(2.14) \qquad (\partial_4 j_i)_j + (\partial_5 s)_{il}(\partial_4 g_l)_j + (\partial_6 s)_{kli}(\partial_4 f_{kl})_j = 0,$$

$$(2.15) \qquad (\partial_5 j_i)_{jk} - (\partial_4 s)_i \delta_{jk} + (\partial_5 s)_{mi}(\partial_5 g_m)_{jk} + (\partial_6 s)_{mni}(\partial_5 f_{mn})_{jk} = 0,$$

$$(2.16) \qquad (\partial_6 j_i)_{jkl} + (\partial_5 s)_{mi}(\partial_6 g_m)_{jkl} + (\partial_6 s)_{mni}(\partial_6 f_{mn})_{jkl} = 0.$$

Finally the residual dissipation inequality can be written in the following form

$$
\begin{aligned}
[\partial_1 j_i &+ (\partial_5 s)_{ji}\partial_1 g_j + (\partial_6 s)_{jki}\partial_1 f_{jk}]e\prime_i + \\
+[(\partial_2 j_j)_i &- \partial_1 s \delta_{ij} + (\partial_5 s)_{kj}(\partial_2 g_k)_i + (\partial_6 s)_{klj}(\partial_2 f_{kl})_i]q_{i\prime j} + \\
+[(\partial_3 j_k)_{ij} &+ (\partial_5 s)_{lk}(\partial_3 g_l)_{ij} + (\partial_6 s)_{lmk}(\partial_3 f_{lm})_{ij}]\Phi_{ij\prime k} + \\
&+ (\partial_2 s)_i g_i + (\partial_3 s)_{ij} f_{ij} \geq 0.
\end{aligned}
$$
$$(2.17)$$

It is easily seen that the Liu system (2.14)-(2.16) is composed by 117 differential equations constraining the set of the 832 partial derivatives of the constitutive functions s, j_i, g_i, f_{ij} with respect to the elements of the constitutive space $(e, q_i, \Phi_{ij}, e\prime_i, q_{i\prime j}, \Phi_{ij\prime k})$. Without some simplifications there is no hope to solve such a system.

In the following sections we are looking for special simplifying assumptions to solve the Liu equations (2.14)-(2.16) and the dissipation inequality (2.17). First we will investigate cases where some of the constitutive functions are assumed to be local.

3. SOLUTIONS OF THE LIU EQUATIONS - LOCALITY ASSUMPTIONS

3.1. **Local state.** Let us start the assumption of the local state in the form that the entropy is independent of the gradients:

$$(3.1) \qquad\qquad s := s(e, q_i, \Phi_{ij})$$

In this case the Liu equations (2.14)-(2.16) are simplified considerably and as a solution we obtain a local entropy current

$$(3.2) \qquad\qquad j_i = j_i(e, q_i, \Phi_{ij}).$$

The dissipation inequality (2.17) simplifies, too

$$
\begin{aligned}
\partial_1 j_i e\prime_i &+ [(\partial_2 j_j)_i - \partial_1 s \delta_{ij}]q_{i\prime j} + (\partial_3 j_k)_{ij}\Phi_{ij\prime k} + (\partial_2 s)_i g_i + (\partial_3 s)_{ij} f_{ij} = \\
&= (j_i)\prime_i - \partial_1 s q_{i\prime i} + (\partial_2 s)_i g_i + (\partial_3 s)_{ij} f_{ij} \geq 0.
\end{aligned}
$$
$$(3.3)$$

3.2. **Local state and local evolution.** Let us investigate now the case when the evolution equations of the internal variables are ordinary differential equations, that is we assume that the constitutive quantities g_i and f_{ij} take the local form:

$$(3.4) \qquad\qquad g_i := g_i(e, q_j, \Phi_{jk})$$

$$(3.5) \qquad\qquad f_{ij} := f_{ij}(e, q_j, \Phi_{jk})$$

Now the dissipation inequality simplifies further the possible constitutive functions, because the coefficients of the derivatives should disappear. These restrictions result in a rather unusual material, since we get

$$(3.6) \qquad j_i = j_i(q_j), \qquad \frac{\partial j_i}{\partial q_j} = \frac{\partial s}{\partial e}\delta_{ij}.$$

On the other hand the dissipation inequality can be written in a force-current form and can be solved for the constitutive functions g_i and f_{ij}

$$(\partial_2 s)_i g_i + (\partial_3 s)_{ij} f_{ij} \geq 0.$$

By (3.6) it follows that the temperature of the material $\frac{\partial s}{\partial e}$ is independent of the internal energy. Such a property is in contrast with the physical reality. We conclude that some nonlocality is necessary in modelling rigid heat conductors.

3.3. Local evolution.

Let us assume now that the evolution of the internal variables is local, but there is no local state, therefore we require (3.4) and (3.5), but (3.1) is not assumed. In this case the entropy current is nonlocal, but the nonlocality is rather reduced. The Liu equations (2.14) and (2.16) give that the entropy current does not depend on the gradients of e and Φ, and (2.15) simplifies to

$$\frac{\partial j_i}{\partial q_{j'k}} = \frac{\partial s}{\partial e_{'i}}\delta_{jk}$$

The nonlocality in the q_i is due to balance form of the evolution equation of the internal energy (2.1). From the above constraint one can easily see, that the entropy current is local if we further assume that the entropy is local in the internal energy,

$$s := s(e, q_i, \Phi_{ij}, q_{i'j}, \Phi_{ij'k}).$$

4. SOLUTIONS OF THE ENTROPY INEQUALITY IN CASE OF LOCAL STATE

As we have seen above, the Liu equations are trivially solvable in the local state. However, the solution of the dissipation inequality can be achieved only with further assumptions. Moreover, there are different assumptions to have physical models, to introduce a suitable gradient dependencies. In Classical Irreversible Thermodynamics [5] and in Rational Thermodynamics [20] the mentioned requirement of nonlocality is achieved by introducing the gradient of temperature (or, equivalently, of the internal energy) into the constitutive space. In Extended Thermodynamics [6, 7] the constitutive space is local but the evolution equations are balances, they have a special nonlocal form. All kind of theories of Extended Thermodynamics resulted in more of less satisfactory models of different phenomena, but they provide different solutions of the entropy inequality.

4.1. Local state and special nonlocal evolution: linear nonlocality.

In this case the evolution equations depend linearly on the gradients:

$$(4.1) \qquad g_i \quad := \quad A_{ij}e_{'j} + B_{ijk}q_{k'j} + C_{ijkl}\Phi_{kl'j},$$
$$(4.2) \qquad f_{ij} \quad := \quad D_{ijk}e_{'k} + E_{ijkl}q_{l'k} + F_{ijklm}\Phi_{lm'k},$$

where $A_{ij}, B_{ijk}, C_{ijkl}, D_{ijk}, E_{ijkl}, F_{ijklm}$ are local constitutive functions.

Now the dissipation inequality (2.17) reduces to a solvable form as

$$(4.3) \qquad (\partial_1 j_i \quad + \quad (\partial_2 s)_j A_{jk} + (\partial_3 s)_{jk} D_{jki})e_{'i}$$
$$(4.4) \qquad +((\partial_2 j_j)_i \quad - \quad \partial_1 s\delta_{ij} + (\partial_2 s)_k B_{kji} + (\partial_3 s)_{kl} E_{klji})q_{i'j}$$
$$(4.5) \qquad +((\partial_3 j_k)_{ij} \quad + \quad (\partial_2 s)_l C_{lkij} + (\partial_3 s)_{lm} F_{lmkij})\Phi_{ij'k} \geq 0.$$

As the quantities in the parentheses are local functions they should be zero respectively. Therefore we get

$$(4.6) \qquad \partial_1 j_i = -(\partial_2 s)_j A_{jk} - (\partial_3 s)_{jk} D_{jki}$$

$$(4.7) \qquad (\partial_2 j_j)_i = \partial_1 s \delta_{ij} - (\partial_2 s)_k B_{kji} - (\partial_3 s)_{kl} E_{klji}$$

$$(4.8) \qquad (\partial_3 j_k)_{ij} = -(\partial_2 s)_l C_{lkij} - (\partial_3 s)_{lm} F_{lmkij}.$$

These equations cannot be solved without any further ado. However, we can see that even if we do no know anything on the entropy current j_i they result in strong correlations on the entropy derivatives and the evolution equation, as the mixed partial derivatives of j_i should be equal. On the other hand let us observe that in this case the entropy production is zero, there is no dissipation.

4.2. **Local state and special local evolution: balance form.** In this case one assumes, that the evolution equations have a special balance form. Therefore there are potentials Q_{ij} and H_{ijk} of the fields $(A_{ij}, B_{ijk}, C_{ijkl})(e, q_i, \Phi_{ij})$ and $(D_{ijk}, E_{ijkl}, F_{ijklm})(e, q_i, \Phi_{ij})$ respectively. The evolution equations can be written as

$$(4.9) \qquad g_i := \partial_1 Q_{ij} e_{'j} + (\partial_2 Q_{ij})_k q_{k'j} + (\partial_3 Q_{ij})_{kl} \Phi_{kl'j},$$

$$(4.10) \qquad f_{ij} := \partial_1 H_{ijk} e_{'k} + (\partial_2 H_{ijk})_l q_{l'k} + (\partial_3 H_{ijk})_{lm} \Phi_{lk'k}.$$

The conditions (4.6)-(4.8) can be written as

$$(4.11) \qquad \partial_1 j_i = -(\partial_2 s)_j \partial_1 Q_{ij} - (\partial_3 s)_{jk} \partial_1 H_{ijk}$$

$$(4.12) \qquad (\partial_2 j_j)_i = \partial_1 s \delta_{ij} - (\partial_2 s)_k (\partial_2 Q_{ij})_k - (\partial_3 s)_{kl} (\partial_2 H_{ijk})_l$$

$$(4.13) \qquad (\partial_3 j_k)_{ij} = -(\partial_2 s)_l (\partial_3 Q_{ij})_{kl} - (\partial_3 s)_{lm} (\partial_3 H_{ijk})_{lm}.$$

As a consequence the above system of equations can be solved, as the entropy current is a potential of the field (q_i, Q_{ij}, H_{ijk}), therefore it can be conveniently written as

$$(4.14) \qquad j_i(e, q_i, \Phi_{ij}) = \tilde{j}_i(q_i, Q_{ij}, H_{ijk}).$$

This fact can be expressed also with differential forms, according to the traditions of thermodynamics

$$(4.15) \qquad \mathrm{d}\tilde{j}_i = \partial_1 s \mathrm{d}q_i + (\partial_2 s)_j \mathrm{d}Q_{ji} + (\partial_3 s)_{jk} \mathrm{d}H_{jki} = \Gamma^1 \mathrm{d}q_i + \Gamma^2_j \mathrm{d}Q_{ji} + \Gamma^3_{jk} \mathrm{d}H_{jki}.$$

The derivatives of the entropy current are identical to the intensives, the derivatives of the entropy functions. However, the variables are different. This form results in serious restrictions of the entropic intensives, and the currents Q_{ij} and H_{ijk} because the mixed second partial derivatives should be equal in these variables, too. Expressed in the basic variables the above requirements are rather ugly

$$(4.16) \qquad (\partial_{22} s)_{kj} \partial_1 Q_{ki} + (\partial_{23} s)_{lkj} \partial_1 H_{lki} = \partial_{11} s \delta_{ij} + (\partial_{12} s)_k (\partial_2 Q_{ik})_j + (\partial_{13} s)_{lk} (\partial_2 H_{lki})_j,$$

$$(4.17) \qquad (\partial_{32} s)_{ljk} \partial_1 Q_{li} + (\partial_{33} s)_{lmjk} \partial_1 H_{lmi} = (\partial_{12} s)_l (\partial_3 Q_{li})_{jk} + (\partial_{13} s)_{lm} (\partial_3 H_{lmi})_{jk},$$

$$(\partial_{31} s)_{kl} \delta_{ij} + (\partial_{32} s)_{mkl} (\partial_2 Q_{im})_j + (\partial_{33} s)_{mnkl} (\partial_2 H_{mni})_j =$$

$$(4.18) \qquad = (\partial_{22} s)_{ml} (\partial_3 Q_{mi})_{jk} + (\partial_{32} s)_{mnl} (\partial_3 H_{mni})_{jk}.$$

The property (4.15) is an important consequence of the balance form of the evolution equations. It is independent of the choice of the basic variables. If one assumes e.g. that the chosen internal variable is the current of the heat flux, $\Phi_{ij} = Q_{ij}$, as it is usual in extended thermodynamics, then the above system of requirements simplifies but does not disappear.

In Rational Extended Thermodynamics it was shown that the above result of the phenomenological theory can be in accordance with the kinetic theory of gases, at least with a classical formulation of kinetic physics. A crucial step in the different systems was the choice of the phenomenological variables (we will see, that all of the currents cannot be chosen as internal variables without any further ado) and the use of source terms in the balances.

4.3. Local state and balance form evolution: isotropy. The system (4.16)-(4.18) does not have a general solution for the currents. Q_{ij} and H_{ijk} cannot be determined by the entropy function in general. Therefore we have lost one of the basic flavors of irreversible thermodynamics, that the requirements of the second law can be exploited constructively to find a the appropriate evolution equations. Now the dissipation inequality was solved, but the evolution equations cannot be determined constitutively.

Jou, Lebon, Mongiovi and Peruzza gave some simplifying conditions to have a solution of the conditions (4.16)-(4.18) on the phenomenological level [21]. They have assumed a local state, balance form evolution equations and a simpler set of variables, they introduced only q_i as an additional variable. Then only condition (4.16) applies in a simplified form as

$$(\partial_{22}s)_{kj}\partial_1 Q_{ki} = \partial_{11}s\delta_{ij} + (\partial_{12}s)_k(\partial_2 Q_{ik})_j.$$

Moreover they have assumed *isotropic materials*, when all scalar valued functions, including the entropy, depend only on $q^2 = q_i q_i$ and the flux of the heat current and the entropy current can be written as

$$(4.19) \qquad Q_{ij} = \beta(e, q^2) + \psi(e, q^2) q_i q_j,$$
$$(4.20) \qquad j_i = \Psi(e, q^2) q_i.$$

Now the requirement (4.15) results in the following system of equations, as (4.2)-(4.4) in [21]

$$(4.21) \qquad \partial_e \Psi = 2\partial_{q^2}s(\partial_e\beta + \partial_e\psi q^2),$$
$$(4.22) \qquad \Psi = \partial_e s + 2\partial_{q^2}s\psi q^2,$$
$$(4.23) \qquad \partial_{q^2}\Psi = 2\partial_{q^2}s\left(\partial_{q^2}\beta + \partial_{q^2}\psi q^2\right).$$

Therefore the entropy current can be written as

$$(4.24) \qquad j_i = \left(\partial_e s + 2\partial_{q^2}s\psi q^2\right)q_i.$$

After further calculations, considering also the requirement (4.16), one can get explicit solutions for the functions β and ψ together with some additional restriction on the form of the entropy function.

5. LOCAL STATE AND SPECIAL ENTROPY CURRENT

A different solution of the dissipation inequality can be given with the help of the entropy current. As in our previous work [8] we consider the local entropy of the form (1.1). This general form is motivated by the requirement of the thermodynamic stability, or, equivalently the requirement of the concavity of the entropy function on the non-equilibrium part of the state space (spanned by q_i and Φ_{ij}). Therefore, m_{ij} and n_{ijkl} are positive definite constitutive functions. Moreover, let us assume the entropy current takes the form (1.2) and let us introduce the convenient notations $\hat{m}, \hat{n}, \tilde{n}, \tilde{m}$ as follows

$$
\begin{aligned}
(\partial_2 s)_i &= -m_{ij}q_j - \frac{1}{2}(\partial_2 m_{lj})_i q_l q_j - \frac{1}{2}(\partial_2 n_{rjkl})_i \Phi_{rj}\Phi_{kl} \\
(5.1) \qquad &= -\hat{m}_{ij}q_j - (\partial_2 \tilde{n}_{jk})_i \Phi_{jk},
\end{aligned}
$$

$$
\begin{aligned}
(\partial_3 s)_{ij} &= -\frac{1}{2}(\partial_3 m_{lk})_{ij}q_l q_k - n_{ijkl}\Phi_{kl} - \frac{1}{2}(\partial_3 n_{lkrs})_{ij}\Phi_{lk}\Phi_{rs} \\
(5.2) \qquad &= -(\partial_3 \tilde{m}_k)_{ij}q_k - \hat{n}_{ijkl}\Phi_{kl}
\end{aligned}
$$

If m_{ij} and n_{ijkl} are constant, then $\hat{m}_{ij} = m_{ij}$, $\hat{n}_{ijkl} = n_{ijkl}$, $(\partial_3 \tilde{m}_k)_{ij} = 0$ and $(\partial_2 \tilde{n}_{jk})_i = 0$.

Let us emphasize again that the entropy, written in the Gyarmati form (1.1) and the entropy current, written in (1.2) are only convenient notations as long as the corresponding inductivities and current multipliers are general constitutive functions.

With the (1.2) form of the entropy current and using the notations of (5.1) and (5.2) the dissipation inequality (3.3) in local state can be written in the following form:

$$[A_{ji'j} - \hat{m}_{ij}g_j - (\partial_3 m)_{kji}f_{ik}]q_i + [A_{ij} - \partial_1 s\delta_{ij}]q_{j'i} +$$

(5.3)
$$[B_{kij'k} - (\partial_2 n)_{kji}g_k - \hat{n}_{ijkl}f_{kl}]\Phi_{ij} + B_{ijk}\Phi_{jk'i} \geq 0.$$

Seemingly the system is a normal force-current system, because the coefficients of the thermodynamic forces $q_i, q_{i,j}, \Phi_{ij}, \Phi_{ijk}$ all contain undetermined constitutive quantities $A_{ij}, g_i, f_{ij}, B_{ijk}$ respectively. However, let us observe that in local state g_i and f_{ij} are nonlocal but all other terms are local in the above inequality. This fact introduces degeneracy since, although the coefficients of the derivatives $q_{i'j}$ and $\Phi_{ij'k}$ cannot disappear, their possible couplings are rather reduced, e.g. B_{ijk} is local, therefore cannot depend on its own force $\Phi_{ij'k}$.

In this degenerate case the solution of the dissipation inequality is not straightforward.

Fortunately we can avoid the treatment of degeneracy, e.g. by assuming that m_{ij} and n_{ijkl} depend only on the internal energy and introducing the form (1.3) of the entropy current with the assumptions $A_{ij} = \partial_1 s\delta_{ij}$ and $B_{ijk} = (\partial_2 s)_k\delta_{ij}$. In this particular case the dissipation inequality reduces to

(5.4)
$$\left((\partial_1 s)_{'i} - m_{ij}g_j - m_{ki}\Phi_{jk'j}\right)q_i + \left(-m_{jl}q_{l'i} - n_{ijkl}f_{kl}\right)\Phi_{ij} \geq 0.$$

This is a force-current system, with the following Onsagerian solution

(5.5)
$$-m_{ij}(\dot{q}_j + \Phi_{kj'k}) + (\partial_1 s)_{'i} = L_{ij}^{11}q_j + L_{ijk}^{12}\Phi_{jk},$$

(5.6)
$$-n_{ijkl}\dot{\Phi}_{kl} - m_{jl}q_{l'i} = L_{ijk}^{21}q_k + L_{ijkk}^{22}\Phi_{kl}.$$

The system above conserves the structure already obtained in [8] in the case of local state. One should emphasize the central role of the invertibility of the matrices m_{ij} and n_{ijkl} in order to obtain the balance form. Such a property is not trivial since there exist real materials for which it is not guaranteed. A classical example is given in [22], where an electric circuit described by dynamic variables is considered.

As it was observed in [8] the second equation is a phenomenological closure of the system (2.1) and (5.5). It is remarkable that we have recovered the usual phenomenological structure of Extended Irreversible Thermodynamics keeping the entropy current of the form (1.3). The same is not true for the constitutive equation (1.2) which, in the case of first order nonlocality, seems to be much too general. However, the compatibility of the entropy current of Verhás (1.3) and the requirements of the balance form (4.15) is valid only with the restriction that the conductivities depend only on the internal energy. The mentioned solution of Jou et. al. (4.24) clearly does not have the Verhás form and indicates the necessity of a more general treatment.

In the next section we study the same problem in the presence of second order nonlocality.

6. SECOND ORDER NONLOCALITY - SOLUTION IN LOCAL STATE

Now we will extend our investigations to consider second order nonlocalities. However, for the sake of simplicity we will investigate only the case of local entropy, i.e. systems in local state. The basic state is spanned by the variables (e, q_i, Φ_{ij}) as previously. However, the constitutive space contains the second order space derivatives and is spanned by $(e, q_i, \Phi_{ij}, e_{'i}, q_{i'j}, \Phi_{ij'k}, e_{'ij}, q_{i'jk}, \Phi_{ij'kl})$. Therefore, in the exploitation of the entropy inequality (2.7) we should consider as constraints the evolution equations (2.1), (2.2) and (2.3), their first prolongations (2.4), (2.5) and (2.6) and also their second prolongations as follows

(6.1)
$$\dot{e}_{'ij} + q_{j'jik} = 0,$$

(6.2)
$$\dot{q}_{i'jk} - g_{i,jk} = 0,$$

(6.3)
$$\dot{\Phi}_{ij'kl} - f_{ij'kl} = 0.$$

Our simplifying condition of local entropy can be written as

$$s = s(e, q_i, \Phi_{ij}).$$

Let us introduce again the Lagrange-Farkas multipliers Γ^1, Γ_i^2, Γ_{ij}^3, Γ_i^4, Γ_{ij}^5 and Γ_{ijk}^6 for the evolution equations (2.1), (2.2), (2.3) and their prolonged forms (2.4), (2.5) and (2.6) respectively. The multipliers Γ_{ij}^7, Γ_{ijk}^8 and Γ_{ijkl}^9 stand for the second prolongations (6.1), (6.2) and (6.3) respectively. The Liu procedure gives

$$
\begin{aligned}
\partial_1 s \dot{e} &+ (\partial_2 s)_i \dot{q}_i + (\partial_3 s)_{ij} \dot{\Phi}_{ij} + \\
+(\partial_1 j_i)e_{\prime i} &+ (\partial_2 j_i)_j q_{j'i} + (\partial_3 j_i)_{jk}\Phi_{jk'i} + (\partial_4 j_i)_j e_{\prime ij} + (\partial_5 j_i)_{jk} q_{j'ki} + \\
+(\partial_6 j_i)_{jkl}\Phi_{jk'li} &+ (\partial_7 j_i)_{jk}e_{\prime ijk} + (\partial_8 j_i)_{jkl}q_{j'kli} + (\partial_9 j_i)_{jklm}\Phi_{jk'lmi} - \\
-\Gamma^1(\dot{e}+q_{i'i}) &- \Gamma_i^2(\dot{q}_i - g_i) - \Gamma_{ij}^3\left(\dot{\Phi}_{ij} - f_{ij}\right) - \Gamma_i^4(\dot{e}_{\prime i} + q_{j'ji}) - \\
-\Gamma_{ij}^5[\dot{q}_{i'j} &- (\partial_1 g_i)e_{\prime j} - (\partial_2 g_i)_k q_{k'j} - (\partial_3 g_i)_{kl}\Phi_{kl'j} - (\partial_4 g_i)_k e_{\prime kj} - \\
&- (\partial_5 g_i)_{kl}q_{k'lj} - (\partial_6 g_i)_{klm}\Phi_{kl'mj} - (\partial_7 g_i)_{kl}e_{\prime klj} - \\
&- (\partial_8 g_i)_{klm}q_{k'lmj} - (\partial_9 g_i)_{klmn}\Phi_{kl'mnj}] \\
-\Gamma_{ijk}^6[\dot{\Phi}_{ij'k} &- (\partial_1 f_{ij})e_{\prime k} - (\partial_2 f_{ij})_l q_{l'k} - (\partial_3 f_{ij})_{lm}\Phi_{lm'k} - (\partial_4 f_{ij})_l e_{\prime lk} - \\
&- (\partial_5 f_{ij})_{lm}q_{l'mk} - (\partial_6 f_{ij})_{lmn}\Phi_{lm'nk} - (\partial_7 f_{ij})_{lm}e_{\prime lmk} - \\
&- (\partial_8 f_{ij})_{lmn}q_{l'mnk} - (\partial_9 f_{ij})_{lmno}\Phi_{lm'nok}] \\
-\Gamma_{ij}^7(\dot{e}_{ij}+q_{k'kij}) &- \Gamma_{ijk}^8\left(\dot{q}_{i'jk} - g_{i'jk}\right) - \Gamma_{ijkl}^9\left(\dot{\Phi}_{ij'kl} - f_{ij'kl}\right) \geq 0.
\end{aligned}
$$

Here ∂_n, $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$ denotes the partial derivatives of the constitutive functions according to the variables of the constitutive space $(e, q_i, \Phi_{ij}, e_{\prime i}, q_{i'j}, \Phi_{ij'k}, e_{\prime ij}, q_{i'jk}, \Phi_{ij'kl})$ respectively. The first set of Liu equations defines the Lagrange-Farkas multipliers as the derivatives of the entropy and gives that the last six multipliers are zero, due to the local state.

$$(6.4) \qquad \qquad \partial_1 s = \Gamma^1,$$
$$(6.5) \qquad \qquad (\partial_2 s)_i = \Gamma_i^2,$$
$$(6.6) \qquad \qquad (\partial_3 s)_{ij} = \Gamma_{ij}^3,$$
$$(6.7) \qquad \qquad \Gamma_i^4 = 0,$$
$$(6.8) \qquad \qquad \Gamma_{ij}^5 = 0,$$
$$(6.9) \qquad \qquad \Gamma_{ijk}^6 = 0,$$
$$(6.10) \qquad \qquad \Gamma_{ij}^7 = 0,$$
$$(6.11) \qquad \qquad \Gamma_{ijk}^8 = 0,$$
$$(6.12) \qquad \qquad \Gamma_{ijkl}^9 = 0.$$

Considering (6.7)-(6.12) the second set of Liu equations are also simple,

$$(6.13) \qquad \qquad (\partial_7 j_i)_{jk} = 0,$$
$$(6.14) \qquad \qquad (\partial_8 j_i)_{jkl} = 0,$$
$$(6.15) \qquad \qquad (\partial_9 j_i)_{jklm} = 0.$$

Therefore the entropy current presents only first order nonlocalities

$$j_i = j_i(e, q_i, \Phi_{ij}, e_{\prime i}, q_{i'j}, \Phi_{ij'k}).$$

Considering the above conditions the dissipation inequality can be written exactly in the same form as it was in case of first order nonlocal constitutive space (3.3):

$$(6.16) \qquad \qquad (j_i)_{\prime i} - \partial_1 s q_{i'i} + (\partial_2 s)_i g_i + (\partial_3 s)_{ij} f_{ij} \geq 0.$$

On the other hand now the constitutive quantities are higher orderly nonlocal, the entropy current is a first order nonlocal quantity and g_i and f_{ij} are second order nonlocal quantities. Let us introduce the same assumptions on the form of the entropy and of the entropy current

as previously with assuming (1.1) and (1.2) with the notations (5.1) and (5.2) but letting the current multipliers A_{ij} and B_{ijk} to contain first order nonlocalities. The dissipation inequality can be written in the same form as above

$$[A_{ji'j} - \hat{m}_{ij}g_j - (\partial_3 m)_{kji}f_{ik}]q_i + [A_{ij} - \partial_1 s\delta_{ij}]q_{j'i} +$$

(6.17) $$[B_{kij'k} - (\partial_2 n)_{kji}g_k - \hat{n}_{ijkl}f_{kl}]\Phi_{ij} + B_{ijk}\Phi_{jk'i} \geq 0.$$

However, in this case it is a non-degenerate force-current system, due to the extension of the constitutive state space. All additive terms contain unknown functions. Therefore one can give an Onsagerian solution as follows

$$-\hat{m}_{ij}g_j - (\partial_3 m)_{ijk}f_{jk} + A_{ji'j} =$$

(6.18) $$L_{ij}^{11}q_j + L_{ijk}^{12}q_{j'k} + L_{ijk}^{13}\Phi_{jk} + L_{ijkl}^{14}\Phi_{jk'l},$$

(6.19) $$A_{ij} - \partial_1 s\delta_{ij} = L_{ijk}^{21}q_k + L_{ijkl}^{22}q_{k'l} + L_{ijkl}^{23}\Phi_{kl} + L_{ijklm}^{24}\Phi_{kl'm},$$

$$-(\partial_2 n)_{kji}g_k - \hat{n}_{ijkl}f_{kl} + B_{kij'k} =$$

(6.20) $$L_{ijk}^{31}q_k + L_{ijkl}^{32}q_{k'l} + L_{ijkl}^{33}\Phi_{kl} + L_{ijklm}^{34}\Phi_{kl'm},$$

(6.21) $$B_{ijk} = L_{ijkl}^{41}q_l + L_{ijklm}^{42}q_{l'm} + L_{ijklm}^{43}\Phi_{lm} + L_{ijklmn}^{44}\Phi_{lm'n}.$$

Here $L^{11}, L^{12}, L^{13}, L^{14}, L^{21}, L^{22}, L^{23}, L^{24}, L^{31}, L^{32}, L^{33}, L^{34}, L^{41}, L^{42}, L^{43}, L^{44}$ are constitutive functions with the suitable definiteness restrictions. Moreover (6.18) and (6.20) are the candidates of balances under suitable restrictions. It is worth noticing, that the current multipliers A_{ij} and B_{ijk} are given explicitly. Therefore, they can be easily eliminated from the above system substituting (6.19) into (6.18) and (6.21) into (6.20). The resulting set of equations is closed and contains first and second order derivatives of the basic state.

$$-\hat{m}_{ij}\dot{q}_j - (\partial_3 m)_{ijk}\dot{\Phi}_{jk} +$$

$$+ [\partial_1 s\delta_{ij} + L_{ijk}^{21}q_k + L_{ijkl}^{22}q_{k'l} + L_{ijkl}^{23}\Phi_{kl} + L_{ijklm}^{24}\Phi_{kl'm}]_{'j} =$$

(6.22) $$= L_{ij}^{11}q_j + L_{ijk}^{12}q_{j'k} + L_{ijk}^{13}\Phi_{jk} + L_{ijkl}^{14}\Phi_{jk'l},$$

$$-(\partial_2 n)_{kji}\dot{q}_k - \hat{n}_{ijkl}\dot{\Phi}_{kl} +$$

$$+ [L_{ijkl}^{41}q_l + L_{ijklm}^{42}q_{l'm} + L_{ijklm}^{43}\Phi_{lm} + L_{ijklmn}^{44}\Phi_{lm'n}]_{'k} =$$

(6.23) $$= L_{ijk}^{31}q_k + L_{ijkl}^{32}q_{k'l} + L_{ijkl}^{33}\Phi_{kl} + L_{ijklm}^{34}\Phi_{kl'm},$$

As one can see, the evolution equations are rather general. They are more general that those given with the help of the entropy currents (1.3). On the other hand the entropy current (4.24) is also a special case of the general (1.2). However, the compatibility to the current potential structure, expressed by (4.15), cannot be expressed explicitly in general. (1.2) is definitely more general regarding the nonlocality, moreover, (4.24) proves, that can be compatible with the potential structure in special cases. However, it definitely restricts the functional form of the current multipliers A_{ij} and B_{ijk}.

Moreover, the potential structure is unavoidable requiring locality and balance form evolution equations. If m_{ij} and n_{ijkl} are constant a balance structure similar to that obtained in [8] is recovered. This fact represents the main effect of the enlargement of the state space since the balance structure is compatible with more general entropy fluxes.

7. Nonlocal second sound: the Guyer-Krumhansl equation

Thermal wave propagation, sometime referred to as second sound, is a low temperature phenomenon which can be observed, for instance, in dielectric crystals such as Sodium Fluoride (NaF) and Bismuth (Bi) [23, 24, 17]. It requires an extension of the classical Fourier's theory in order to remove the paradox of infinite speed of propagation of thermal disturbances [25]. Phonon gas hydrodynamics [26, 27] supplies a satisfactory explanation of heat transport at low temperature. Phonons are quasi-particles which obey the Bose-Einstein statistics. In a solid crystal they form a rarefied gas whose kinetic equation can be derived similarly to that

of an ordinary gas. Phonons may interact among themselves and with lattice imperfections through two different types of processes:

i) Normal-(N) processes, that conserve the phonon momentum;

ii) Resistive-(R) processes, in which the phonon momentum is not conserved.

The frequencies ν_N and ν_R of normal and resistive processes determine the characteristic relaxation times $\tau_N = \frac{1}{\nu_N}$ and $\tau_R = \frac{1}{\nu_R}$. Diffusive processes take over when there are many more R-processes than N-processes. If instead there are only few R-processes and many more N-processes, then a wave like energy transport may occur.

Such a phenomenology is satisfactorily described by the Guyer-Krumhansl equation [14], eq. (57) (see also [28, 15]).

$$(7.1) \qquad \dot{q}_i + \frac{1}{\tau_R}q_i + \frac{1}{3}c^2 c_V T_{,i} = \frac{1}{5}\tau_N c^2 \left(q_{i'jj} + 2q_{j\,'ji}\right),$$

where $T = \hat{T} - T_0$ is the temperature variation (T is the temperature and T_0 is the average temperature). c_V is the specific heat and c means the Debye phonons velocity. Such an equation, which generalizes the Maxwell-Cattaneo-Vernotte equation [25]

$$(7.2) \qquad \tau_R \dot{q}_i + q_i = -\tau_R \frac{1}{3}c^2 c_V T_{,i},$$

was the first in the literature to include both relaxation times. It can be obtained by the linearized Boltzman equation for phonons in the Debye approximation, if one maintains terms $O(\tau_N)$ [14]. The material coefficients τ_N, τ_R and c_V are all depend on the temperature. According to experimental observations [16] one can get

$$c_V = aT^3, \qquad \tau_R = d\,e^{-\alpha/T}, \qquad \tau_N = bT^{-m}.$$

Here a, b, d, α are constant coefficients and $m \in \{3, 4, 5\}$ depending on the material. Let us remark that the last function, a temperature dependent τ_N is clearly contradicts to the assumptions made by Guyer and Krumhansl [14] p771, and did not consider, that (7.1) is a linearized equation.

In Extended Thermodynamics the Guyer-Krumhansl equation can be recovered in a 4-field theory provided one assumes weakly nonlocal constitutive equations for the internal energy e and for the momentum flux [17].

However, such an approach seems to be questionable because:

i) a nonlocal constitutive space is in contrast with the basic assumptions of Extended Thermodynamics.

ii) a nonlocal internal energy does not assure that the specific heat $c_V = \frac{\partial e}{\partial T}$ is positive.

iii) a nonlocal internal energy would modify also the energy balance.

The last observation was pointed also by Dreyer and Struchtrup and they suggested to consider higher order moments [6, 7]. The other important question is the temperature dependency of the coefficients. A phenomenological theory cannot predict the exact form to the constitutive functions, but gives restrictions and interrelations. These restrictions on the temperature dependencies are frequently treated rather loosely to get the compatibility with the *linearized* kinetic theory [14, 6].

These problems do not arise in the present theory. In fact, we can obtain the equation (7.1) by considering a 4-field model with second order nonlocality, namely the balance equation (6.22) with $\Phi_{ij} = 0$ and m_{ij} depending only on the internal energy. It yields

$$(7.3) \qquad -m_{ij}\dot{q}_j + \left[\partial_1 s\delta_{ij} + L^{21}_{ijk}q_k + L^{22}_{ijkl}q_{k'l}\right]_{,j} = L^{11}_{ij}q_j + L^{12}_{ijk}q_{j'k}.$$

It is convenient to choose as equilibrium thermodynamic variable the absolute temperature T_a instead of the internal energy e. Let us assume the constitutive equations $L^{21}_{ijk} = L^{12}_{ijk} = 0$. we may exploit the thermodynamic relation $\frac{1}{T_a} = \frac{\partial s}{\partial e}$ and introduce the temperature perturbation

T by $T_a = T_0 + T$, where T_0 is a background (average) temperature. Now, equation (7.3) reduces to

$$(7.4) \qquad m_{ij}\dot{q}_j + L_{ij}^{11}q_j + \frac{1}{T^2}T_{,i} = L_{ijkl}^{22}q_{k'lj}.$$

Here $\mathbf{m}$, $\mathbf{L}^{11}$ and $\mathbf{L}^{22}$ can depend on the background temperature T_0. Finally, equation (7.1) is easily obtained by (7.4) under the further constitutive assumptions

$$(7.5) \qquad m_{ij} = \frac{3}{c^2 a T^5}\delta_{ij},$$

$$(7.6) \qquad L_{ij}^{11} = \frac{3}{c^2 a T^5}e^{\frac{\alpha}{T}}\delta_{ij},$$

$$(7.7) \qquad L_{ijkl}^{22} = \frac{3b}{5aT^{5+m}}\delta_{ij}\delta_{kl}.$$

That way, the Guyer-Krumhansl theory may be obtained in a classical macroscopic framework. Let us remark, that the previous phenomenological linearization is only a last step in the calculation based on the linearization of the Boltzmann equation (eq. (57) to eq. (59) in [14]). A similar result has been obtained in [29].

8. Conclusions

We have shown that Classical Irreversible Thermodynamics supplemented with dynamic degrees of freedom is consistent with the idea of higher order fluxes as independent thermodynamic variables. In such a framework the balance form of the evolution equations can be obtained under suitable constitutive assumptions. The locality of the entropy density, i.e. the local state assumption, plays a central role. Moreover, the nonlocality of the constitutive functions results in a wide class of materials including the classical Cattaneo's and Guyer-Krumhansl heat conductors, which normally are derived from kinetic theory. Let us observe that the model above encompasses all extended thermodynamic models, since the balance structure represents a very particular form of nonlocal evolution equations (2.2) and (2.3).

The figure below shows the connections between the constitutive assumptions and the thermodynamic restrictions, as far as the locality and nonlocality are concerned. The last column refers to the corresponding section of the paper. L_A denotes a locality assumption. The double head arrows denote specific interrelations, e.g. the black double head arrows in the first row denote equations (2.14)-(2.16), that give conditions between the entropy and entropy current functions and evolution equations. The fifth row refers to Rational Extended Thermodynamics and the sixth row to the specific solution of the potential restriction given by Jou, Lebon, Mongiovi and Peruzza. As in Extended Thermodynamics the dissipations inequality is fulfilled as an equality, we cannot conclude anything on the dissipative constitutive functions (that is why there are no white arrows in these rows). Let us observe, that assumptions on the form of the entropy current makes possible to build all requirements of the Second Law into the evolution equations in general (last two rows). In this case the Second Law become a material property, satisfied independently of the initial conditions and the resulted constitutive functions depend only on the material.

It is worth noticing that the potential form of the entropy current and the balance structure is compatible with the general form of the entropy current, such as that proposed by Nyíri, which yields (1.3) when $A_{ij} = \partial_1 s\delta_{ij}$ and $B_{ijk} = (\partial_2 s)_k\delta_{ij}$. In this case the balance structure results in restrictions for the current multipliers from the potential requirement (4.15).

We obtained closure relations both with local and nonlocal entropy current and in the first case the closure for the highest order flux was an ordinary differential equation. The obtained thermodynamic closure of the hierarchical structure stresses some deeper relations between the thermodynamic and the more detailed kinetic structure, similar that was recognized e.g. in [30].

We reinspected the thermal wave propagation at low temperature and proved that the Guyer-Krumhansl theory of second sound can be derived in the framework of macroscopic nonlocal irreversible thermodynamics.

9. ACKNOWLEDGEMENTS

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	Entropy (s)	Entropy flux (j)	Evolution (f,g)	
Local	L_A		L_A (trivial)	(3.2)
	$L(e)$ ←	$L(e,\Phi)$ ←	L_A	(3.3)
First order nonlocal	nL ⇔	nL ⇔	nL	(2)
	L_A →	L (spec.) ⇔	nL_A (linear)	(4.1)
	L_A →	L (potential) ←	nL_A (balance)	(4.2)
	$L_A(e,q)$ →	j(s) (JLMP solution) ←	$nL_A(e,q)$ (balance)	(4.3)
	L_A →	L ⇒(Verhás)	nL (solution)	(5)
Second order nonlocal	L_A →	nL_1 ⇒(Nyíri)	nL_2 (solution)	(6)

FIGURE 1. Notations: L - local, nL - nonlocal, L(e) - local in the variable e, L_A - assumed locality, L_i - i-th order nonlocal, Black arrow - consequence of the Liu equations, White arrow - consequence of the dissipation inequality

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